The asymptotic distribution of the partial attributable risk in cross-sectional studies

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Summary. The partial attributable risk has been introduced as a tool for partitioning the responsibility for causing an adverse event between various risk factors. It has arisen from epidemiology, but is also a valid general risk allocation concept, which can for example be applied to data from customer satisfaction surveys. So far, a variance formula for the partial attributable risk has been missing so that confidence intervals were not directly available. This paper provides the asymptotic normal distribution for the partial attributable risk when determined from a cross-sectional study.

Key words: partial attributable risk, partial attributable difference, asymptotic distribution, implicit function theorem, risk allocation, variance estimation

1 INTRODUCTION

In various fields of application, responsibility for occurrence of an adverse event has to be rationally partitioned between several possible causes. The demand for such risk partitioning may be driven by the wish for prioritizing intervention actions or assigning appropriate shares of legal or financial responsibility to different organizations. A prominent application field is epidemiology, where the prevalence of a disease in the population is to be attributed to various possible causation factors. Here, the partial attributable risk was developed as a very natural answer to the risk partitioning task (Eide and Gefeller, 1995). So far, a variance estimate for the partial attributable risk has been missing. For cross-sectional studies with binary possible causes, this very severe limitation will be remedied in this paper. The method is applied to market research data, where the risk for customer dissatisfaction is to be partitioned between different types of troubles that may have occurred on the customers' vehicles (Grömping and Weimann, 2003).

In the following, the ideas behind the partial attributable risk are outlined. For the case of a single possible cause only, the attributable risk has been developed in epidemiology as a measure of the proportion of diseased in the population that can be attributed to that cause, e.g. the proportion of lung cancer cases that can be attributed to smoking. Generally, in a cross-sectional study with a binary variable Y denoting an adverse event like occurrence of a disease (1=event, 0=no event) and a binary cause variable X (1=exposed to possible cause, 0=not exposed), the attributable risk is defined as

$$AR = \left\{P(Y=1) - P(Y=1 \mid X=0)\right\} / P(Y=1)\,. \tag{1}$$

AR can be interpreted as the proportion of adverse events that would not occur if the possible cause X could be completely eliminated. Similarly, when asking the question how many percentage points of adverse events could be avoided by eliminating all possible causes, the answer can be found by looking at the numerator of (1) only. We coin the term Attributable Difference (AD) for this quantity, as we are not aware of any commonly used term. This quantity is not widely used in epidemiology, even though it answers another

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legitimate question. Attributable risk and attributable difference can be estimated by replacing the theoretical probabilities in (1) with their empirical equivalents.

Now the question of partitioning the risk of an adverse event into contributions from different possible causes is considered. First of all, the overall contribution of all possible causes – the so-called common attributable risk – is determined by applying (1) with $X = 0$ denoting the absence of all possible causes. This is then the quantity to be partitioned among the different possible causes. One could come up with the idea to simply apply (1) to each possible cause separately. However, this approach will often yield individual AR estimates that sum to a lot more than the common attributable risk, often even to more than 100% (cf. e.g. Walter 1980). This implausible behavior of the separate simple ARs is due to correlation between the occurrences of different possible causes (for a more thorough discussion, cf. e.g. Baur 2000). As long as only one of the possible causes is of primary interest, it is possible to use the so-called adjusted attributable risk, which is the fraction of dissatisfied attributable to a particular possible cause after adjusting for all other possible causes (Walter 1980, 1983). In customer satisfaction applications, we want to treat several possible causes on equal footing. For this purpose, the adjusted attributable risk is not a good solution: if we just compare the adjusted attributable risk of each possible cause adjusted for all the others, we again encounter the problem that the sum of all adjusted attributable risks is an implausible quantity. It is possible to calculate a sequence of adjusted attributable risks that can be used for determining so-called sequential attributable fractions (which are not attributable risks themselves) that sum to the common attributable risk. However, these are sequential in the sense that they depend on the order in which the possible causes are considered (cf. e.g. Gefeller et al. 1998), which is undesirable. As a solution to these issues, the partial attributable risk (PAR) has been developed (Eide and Gefeller, 1995). The basic idea behind the PAR is simple and has already been proposed in similar contexts by Cox (1985) and Kruskal (1987): acknowledging the fact that the sequential attributable fractions depend on the sequence of adjustment, the PAR is simply an average over all possible adjustment sequences. The PAR has arisen from epidemiology, but is a good general concept of risk allocation: Land and Gefeller (1997) have shown that the PAR is unique in that it is the only risk allocation functional that satisfies several desirable criteria simultaneously. A computation-friendly formulation of the PAR will be introduced in the following section, based on Weimann's Diplomarbeit (Baur, 2000). Additionally, we will also provide the percentage points of satisfaction that can be regained by removing a certain possible cause, which we call partial attributable difference (PAD). Again, this term is coined here, as far as we are aware.

Up to now, a variance formula for the PAR has been missing, and methods like the bootstrap are quite cumbersome (or fail), as for a moderate number of possible causes to be considered on equal footing, calculation of even one set of partial attributable risks takes quite some time. This paper provides the asymptotic distribution of both PAR and PAD for cross-sectional studies (like customer surveys). Attention is restricted to binary possible causes. The derivation of the variance formula relies heavily on work by Benichou and Gail (1989), who state a delta method for implicitly defined estimators. Basu and Landis (1995) article on the variance formula for model-based attributable risks for cross-sectional studies has also been useful.

Section 2 formulates the partial attributable risk (PAR) and the analogous partial attributable difference (PAD) for cross-sectional studies in a form suitable for deriving the variance formula. The estimates can be both model-based and model-free, where the latter is feasible for a relatively small number of possible causes only. Section 3 provides a version of Benichou's and Gail's (1989) result as used in this paper and derives the variance formula for the model-based PAR and PAD. Section 4 provides the analogous results for the model-free scenario. Section 5 gives an example on automotive customer satisfaction data. The discussion points to areas for further research.

2 THE PARTIAL ATTRIBUTABLE RISK FOR CROSS-SECTIONAL STUDIES

Let us consider a situation with $L\geq 2$ binary random variables $X_1,...,X_L$, that take the role of possible causes (1=possible cause present, 0=possible cause absent) and are to be considered on equal footing. Let a stratum C_i, $i=0,\ldots,2^L-1$, be defined as a particular pattern of presence and absence of possible causes, C_0 denoting no TGW at all, C_{2^L-1} denoting all possible causes present. For notational and computational convenience, intermediate strata are ordered according to the system of binary numbers (although the order would not matter, as long as the index i is used compatibly in all formulae). Let us again assume a binary dissatisfaction variable Y as before, where the distribution of Y depends on the stratum C_i.

Without loss of generality, X_1 is singled out as the possible cause of interest. (If interest is in a different possible cause, simply rearrange the X's.) According to Baur (2000; based on Wille and Gefeller 1996), the partial attributable risk for X_1 can then be written as

$$PAR^1 = \frac{1}{L!P(Y=1)} \sum_{i=0}^{2^{L-1}-1} P\left(C_i^{L-1}, X_1=1\right) \sum_{j=0}^{2^{L-1}-1} \left(L-j^+-1\right)!\left(j^+\right)!\left\{P(Y=1 \mid C_i^{L-1} \circ C_j^{L-1}, X_1=1) - P(Y=1 \mid C_i^{L-1} \circ C_j^{L-1}, X_1=0)\right\} \tag{2}$$

where C_i^{L-1} denotes the i-th stratum in the $L-1$ dimensions corresponding to $X_2,...,X_L$, $C_i^{L-1} \circ C_j^{L-1}$ denotes the stratum with ones where both the i-th and the j-th stratum have a one and zeroes elsewhere, and j^+ denotes the number of possible causes present in C_j^{L-1}. Analogously, the partial attributable difference for X_1 can then be written as

$$PAD^1 = \frac{1}{L!} \sum_{i=0}^{2^{L-1}-1} P\left(C_i^{L-1}, X_1=1\right) \sum_{j=0}^{2^{L-1}-1} \left(L-j^+-1\right)!\left(j^+\right)!\left\{P(Y=1 \mid C_i^{L-1} \circ C_j^{L-1}, X_1=1) - P(Y=1 \mid C_i^{L-1} \circ C_j^{L-1}, X_1=0)\right\} \tag{3}$$

The stratum probabilities P(C_i^{L-1} , X_1=1) can be directly estimated from the sample. If all 2^L possible strata occur in the data, the conditional probabilities in formulae (2) and (3) can also be directly estimated by the stratum specific empirical dissatisfaction probabilities. In case of many possible causes, there will most likely be quite a few empty strata. For example, with 12 possible causes, there are 4096 strata, some of which have to be empty if the sample is not very large. In this case, the conditional dissatisfaction probabilities in formulae (2) and (3) can be estimated using a logistic regression model

$$P(Y=1|C_i) = E(Y|C_i) = \mu_i(\boldsymbol{\beta}) = \exp(\mathbf{x}_i\boldsymbol{\beta})/\{1+\exp(\mathbf{x}_i\boldsymbol{\beta})\} \tag{4}$$

with $\mathbf{x}_i$ denoting the 1xq design row for stratum i and $\boldsymbol{\beta}$ denoting a qx1 vector of unknown regression parameters. The marginal probability P(Y=1) can be estimated from stratum probabilities and conditional probabilities using the theorem of total probabilities.

For discussing estimation of (2) and (3), let us denote the empirical dissatisfaction probabilities in stratum C_i as $\bar{y}_i$ and the empirical probabilities of a customer belonging to stratum C_i as $p_i=n_i/N$, $i=0,\ldots,2^L-1$ (where N denotes the total number of observations, n_i the number of observations in C_i). Thus, the actual estimate of the PAR for X_1 is either directly a function of the $\bar{y}_i$ and p_i, or a function of **b** and the p_i, where **b** is the maximum likelihood (ML) estimate for $\boldsymbol{\beta}$ in the logistic model (4). For both situations, the variance formulae for PAR and PAD will be provided. In the following, let $\wp_i$ denote the p_i in its role as an estimate for the probability of stratum C_i, and let $\boldsymbol{\wp}$ denote the $2^L\times 1$ vector of the $\wp_i$. This notational sophistication will prove helpful for applying the delta method for implicit functions. Furthermore, let $\boldsymbol{\pi}=(\pi_i)_{i=0,\ldots,2^L-1}$ denote the vector of true stratum probabilities and $\mathbf{p}=(p_i)_{i=0,\ldots,2^L-1}$ the vector of empirical stratum probabilities. Finally, the superscript T denotes a transposed matrix, est(•) denotes an estimate, asyCov(•) denotes an asymptotic variance-covariance matrix, and asyCov(•,•) denotes an asymptotic covariance matrix between two vectors.

3 THE VARIANCE FORMULA FOR MODEL-BASED PAR AND PAD

Writing est(PAR^I) as exp[ln{est(PAR^I)}], the linear Taylor expansion of the obvious estimate of (2) w.r.t. **b** and $\wp$ is

$$PAR^I + PAR^I\, \mathbf{A}_{PAR,mod}^{T} \{(\mathbf{b}^T, \wp^T)^T - (\boldsymbol{\beta}^T, \boldsymbol{\pi}^T)^T\}, \tag{5}$$

where $\mathbf{A}_{PAR,mod}$ is the column vector of derivatives of ln{est(PAR^I)} w.r.t. the components of **b** and $\wp$, evaluated at the true parameter point $\mathbf{b}=\boldsymbol{\beta}$, $\wp=\boldsymbol{\pi}$. The difference (3) can be treated in complete analogy, with the vector $A_{PAD,mod}$ instead of $\mathbf{A}_{PAR,mod}$. In the following, all results are stated for the PAR only. Results for the PAD are completely analogous, with the terms indexed with PAR replaced as appropriate.

Assuming that **b** and $\wp$ follow a common multivariate normal distribution (as will be demonstrated below), $N^{1/2}$ {est(PAR^I) – PAR^I} is also asymptotically normal with expectation 0 and the asymptotic variance

$$(PAR^I)^2 \mathbf{A}_{PAR,mod}^{T}\, \text{asyCov}\{N^{1/2}\, (\mathbf{b}^T, \wp^T)^T\}\, \mathbf{A}_{PAR,mod}. \tag{6}$$

According to ML theory, $N^{1/2}(\mathbf{b}-\boldsymbol{\beta})$ is asymptotically normal with expectation $\mathbf{0}_q$ and the ML-derived covariance matrix $\left[\sum_{i=0}^{2^L-1} \pi_i \mu_i(\boldsymbol{\beta})\{1-\mu_i(\boldsymbol{\beta})\}\mathbf{x}_i^T \mathbf{x}_i\right]^{-1}$. Also, $N^{1/2}(\wp-\boldsymbol{\pi})$ is asymptotically normal with the multinomial-ML-derived covariance matrix $\boldsymbol{\Sigma}(1,\boldsymbol{\pi})$, where $\boldsymbol{\Sigma}(1,\boldsymbol{\pi})$ denotes the covariance matrix of a multinomial distribution with probabilities collected in $\boldsymbol{\pi}$ and N=1. Using the implicit function theorem as formulated in Benichou and Gail (1989) will demonstrate the common multivariate normal distribution of **b** and $\wp$ and yield the covariance matrix between them, so that (6) can be used. It will be seen that, for a correctly specified logistic model, **b** and $\wp$ are asymptotically uncorrelated.

For the asymptotic covariance matrix between **b** and $\wp$, the following lemma will be used which is a reformulation of Remark 2 in Benichou and Gail (1989):

Lemma 1 Let the $k\times 1$ random vector $\mathbf{Y}_N$ be such that the normalized vector $N^{1/2}(\mathbf{Y}_N - \boldsymbol{\nu}_y)$ converges to a multivariate normal distribution with expectation $\mathbf{0}_k$ and the $k\times k$ covariance matrix $\mathbf{V}$ for $N \to \infty$. If two vectors of random variables, $\mathbf{c}_N$ of dimension q_1 and $\mathbf{d}_N$ of dimension q_2, are separately defined by implicit functions $g_1(\mathbf{c}_N, \mathbf{Y}_N) = \mathbf{0}_{q_1}$ and $g_2(\mathbf{d}_N, \mathbf{Y}_N) = \mathbf{0}_{q_2}$, then under standard regularity conditions, as stated e.g. in Benichou and Gail (1989, Corollary 1), $N^{1/2}\{(\mathbf{c}_N^T, \mathbf{d}_N^T)^T - (\boldsymbol{\gamma}^T, \boldsymbol{\delta}^T)^T\}$ converges to a multivariate normal distribution with expectation $\mathbf{0}_{q_1+q_2}$, where $\boldsymbol{\gamma}$ is the solution for $\mathbf{c}$ of $g_1(\mathbf{c}, \boldsymbol{\nu}_y) = \mathbf{0}_{q_1}$ and $\boldsymbol{\delta}$ is the solution for $\mathbf{d}$ of $g_2(\mathbf{d}, \boldsymbol{\nu}_y) = \mathbf{0}_{q_2}$. The q_1xq_2 upper right block of the covariance matrix is given by

asyCov($N^{1/2}\mathbf{c}_N$, $N^{1/2}\mathbf{d}_N$) =

$$\lim_{N\to\infty} \left(\frac{\mathrm{d}\, g_1}{\mathrm{d}\, \mathbf{c}_N}\right)^{-1} \left(\frac{\mathrm{d}\, g_1(\boldsymbol{\gamma},\boldsymbol{\delta};\mathbf{Y}_N)}{\mathrm{d}\, \mathbf{Y}_N}\right) \mathbf{V} \left(\frac{\mathrm{d}\, g_2(\boldsymbol{\gamma},\boldsymbol{\delta};\mathbf{Y}_N)}{\mathrm{d}\, \mathbf{Y}_N}\right)^{T} \left(\frac{\mathrm{d}\, g_2}{\mathrm{d}\, \mathbf{d}_N}\right)^{-1}, \tag{7}$$

where the limit is obtained by evaluating all derivatives at $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ and $\boldsymbol{\nu}_y$.

In our case, $\mathbf{c}_N$=**b** and $\mathbf{d}_N=\wp$, and $\mathbf{Y}_N$ can be chosen as the $2*2^L$ vector consisting of the proportions $p_i=n_i/N$ of elements in stratum i, $i=0,\ldots,2^L-1$, and the proportions $\bar{y}_i$, $i=0,\ldots,2^L-1$ of dissatisfied customers in stratum i. If p_i=0, the proportion $\bar{y}_i$ has to be arbitrarily fixed at some value: we decide on assigning the value 0 for these cases. g_1(•) is the derivative of the logistic regression log-likelihood, and g_2(•) is an N-th of the derivative of the multinomial log-likelihood, incorporating the Lagrange summand for the restriction that the probabilities sum to 1 (note: we need the fact that 0^0=1 for covering strata with π_i=0):

$$g_1(\mathbf{b};p_i,\ \bar{y}_i) = \sum_{i=0}^{2^L-1} p_i \mathbf{x}_i^{\mathrm{T}} \{\bar{y}_i - \mu_i(\mathbf{b})\}, \tag{8a}$$

$$g_2(\wp;p_i,\ \bar{y}_i) = \left(\begin{cases} p_i/\wp_i - 1 & p_i > 0 \\ 0 & p_i = 0 \end{cases} \right)_{i=0,\ldots,2^L-1}, \tag{8b}$$

with $\mu_i(\boldsymbol{\beta})$ denoting the probability for dissatisfaction in stratum i, as modeled by the logistic model (4). Note that p_i coincides with $\wp_i$, i.e. the element p_i of $\mathbf{Y}_N$ is the solution for $\wp_i$ in $g_2(\wp, \mathbf{Y}_N)=\mathbf{0}$. Although these are the same when calculated, they play different roles in differentiation. The derivatives of $g_1(\bullet)$ and $g_2(\bullet)$ w.r.t. the data vector $\mathbf{Y}_N$ have been calculated in Appendix 1 ((A.1) to (A.4)), as well as the derivatives of $g_1(\bullet)$ w.r.t. **b** and $g_2(\bullet)$ w.r.t. $\wp$ ((A.5) and (A.6)).

<u>Remark 2</u> *(7) and the regularity conditions in Benichou and Gail (1989) require that* $\mathrm{d}g_1/\mathrm{d}\mathbf{c}_N$ *and* $\mathrm{d}g_2/\mathrm{d}\mathbf{d}_N$ *are invertible. Formally, this requirement is violated for* $\mathrm{d}g_2/\mathrm{d}\mathbf{d}_N$ *because of our chosen notation: We keep non-existing strata – i.e. strata with* $\pi_i=0$ *– in all sums (their contribution is always 0) in order to work with the convenient binary digit-equivalent stratum numbering and avoid a special notation for excluding those strata. However, the violation is formal only. Replacing the critical inverse matrix by the Moore-Penrose inverse (denoted by the superscript "+" instead of the superscript "–1"), and applying Benichou's and Gail's regularity conditions to the matrices reduced by rows and columns corresponding to non-existing strata eliminates all issues.*

With $N\rightarrow\infty$, $\mathbf{Y}_N$ approaches $\mathbf{v}_y=(\boldsymbol{\pi}^{\mathrm{T}}, \boldsymbol{\mu}^{\mathrm{T}})^{\mathrm{T}}$, where μ_i=E($Y|C_i$) for π_i>0, μ_i=0 for π_i=0. Furthermore, NCov($\mathbf{Y}_N$) approaches a block-diagonal matrix, as the $\bar{y}_i$ and the p_i are asymptotically uncorrelated with each other. (Asymptotics are needed here for removing bias and correlations arising from empty strata.) The marginal covariance matrix of the vector **p** is the covariance matrix $\boldsymbol{\Sigma}(1, \boldsymbol{\pi})/N$, i.e. an N-th of a multinomial(1, $\boldsymbol{\pi}$) covariance matrix. Hence the upper left block of **V** is $\boldsymbol{\Sigma}(1, \boldsymbol{\pi})$. The conditional distribution of $\bar{y}_i$ given p_i can be simply derived from the binomial distribution (or Dirac(0) for p_i=0), and asymptotically the $N^{1/2}\bar{y}_i$ are uncorrelated with each other with variance $\mu_i(\boldsymbol{\beta})\{1-\mu_i(\boldsymbol{\beta})\}/\pi_i$ for strata with π_i>0 and 0 for strata with π_i=0. However, the covariance matrix of the $\bar{y}_i$ is not needed for applying (7), as the derivative (A.2) is 0. Hence, the asymptotic distribution of ($N^{1/2}\mathbf{b}_N$, $N^{1/2}\wp_N$) can be determined from Lemma 1, and asyCov($N^{1/2}\mathbf{b}_N$, $N^{1/2}\wp_N$) is given as

$$\lim_{N\to\infty} \left\{ \left(\frac{\partial g_{1;i}}{\partial \mathbf{b}_j} \right)_{i,j;\mathbf{b}=\boldsymbol{\beta}} \right\}^{-1} diag\left(\frac{\partial g_1(\boldsymbol{\beta},\boldsymbol{\pi},\mathbf{Y}_N)}{\partial p_i} \right)_{\mathbf{p}=\boldsymbol{\pi}} \boldsymbol{\Sigma}(1,\boldsymbol{\pi}) diag\left(\frac{\partial g_2(\boldsymbol{\beta},\boldsymbol{\pi},\mathbf{Y}_N)}{\partial p_i} \right)_{\mathbf{p}=\boldsymbol{\pi}}^T \left\{ \left(\frac{\partial g_{2;i}}{\partial \wp_j} \right)_{j,i;\wp=\boldsymbol{\pi}} \right\}^{+}$$

$$= -\lim_{N\to\infty} \left\{ \left(\frac{\partial g_{1;i}}{\partial \mathbf{b}_j} \right)_{i,j;\mathbf{b}=\boldsymbol{\beta}} \right\}^{-1} diag\left(\frac{\partial g_1(\boldsymbol{\beta},\boldsymbol{\pi},\mathbf{Y}_N)}{\partial p_i} \right)_{\mathbf{p}=\boldsymbol{\pi}} \boldsymbol{\Sigma}(1,\boldsymbol{\pi})\,. \tag{9}$$

Basu and Landis (1995) already applied the delta method according to Lemma 1 to the attributable risk, and proposed to consistently estimate the covariance matrix between the $\mathbf{b}_N$ and $\wp_N$. If the logistic model (4) correctly models μ_i = E($Y|C_i$), a deeper look at (9) shows that the limit is in fact a matrix of zeroes, as (A.3) is 0 for correctly modeled expectations. Thus, instead of using a consistent estimate, we can use the limit directly. Consequently, splitting the vector $\mathbf{A}_{\mathrm{PAR,mod}}$ into parts for $\wp$ and **b,** formula (6) simplifies to

$$\left(\mathrm{PAR}^{I}\right)^{2}\left[\mathbf{A}_{\mathrm{PAR,mod};\mathbf{b}}{}^{\mathrm{T}}\left\{\sum_{i=0}^{2^{L}-1}\pi_{i}\mu_{i}(\boldsymbol{\beta})\left(1-\mu_{i}(\boldsymbol{\beta})\right)\mathbf{x}_{i}{}^{\mathrm{T}}\mathbf{x}_{i}\right\}^{-1}\mathbf{A}_{\mathrm{PAR,mod};\mathbf{b}}\right.$$
$$\left.+\,\mathbf{A}_{\mathrm{PAR,mod};\wp}{}^{\mathrm{T}}\boldsymbol{\Sigma}(1,\boldsymbol{\pi})\mathbf{A}_{\mathrm{PAR,mod};\wp}\right] \tag{10}$$

$$=\left(\mathrm{PAR}^{I}\right)^{2}\left[\mathbf{A}_{\mathrm{PAR,mod};\mathbf{b}}{}^{\mathrm{T}}\left\{\sum_{i=0}^{2^{L}-1}\pi_{i}\mu_{i}(\boldsymbol{\beta})\left(1-\mu_{i}(\boldsymbol{\beta})\right)\mathbf{x}_{i}{}^{\mathrm{T}}\mathbf{x}_{i}\right\}^{-1}\mathbf{A}_{\mathrm{PAR,mod};\mathbf{b}}\right.$$
$$\left.+\sum_{i=0}^{2^{L}-1}\pi_{i}\mathbf{A}_{\mathrm{PAR,mod},\wp;i}^{2}-\left\{\mathbf{A}_{\mathrm{PAR,mod},\wp}{}^{\mathrm{T}}\boldsymbol{\pi}\right\}^{2}\right],$$

where $\mathbf{A}_{\mathrm{PAR,mod}}$ is given in Appendix 2 (as well as its analogue $\mathbf{A}_{\mathrm{PAD,mod}}$). The asymptotic variance (10) can be consistently estimated in the obvious way.

In the following section, the model-free version for situations with few possible causes is investigated. Although it is also possible to handle the model-free situation by a full logistic model with all possible interaction effects, brief coverage of the direct model-free approach is considered helpful.

4 THE VARIANCE FORMULA FOR MODEL-FREE PAR AND PAD

For this section, it is assumed that $\pi_i>0$ for all strata, i.e. for sufficiently large N all strata are non-empty. Furthermore it is assumed that $\mathrm{P}(Y=1|C_i)$ is neither 0 nor 1 for all i. Instead of applying a logistic regression model, we now estimate $\mathrm{P}(Y=1|C_i)$ by $\bar{y}_i$. Again, we need a notational distinction between the data $\bar{y}_i$ and the same quantity in its role as an estimate for $\mu_i=\mathrm{P}(Y=1|C_i)$, which will be denoted by m_i (combined into a 2^L-vector $\mathbf{m}$). The estimating function (8a) has to be reformulated:

$$g_1(\mathbf{m};p_i,\ \bar{y}_i)=\left\{p_i\left(\bar{y}_i-m_i\right)\right\}_{i=0,\ldots,2^{L}-1}. \tag{8a*}$$

In all other formulae, $\mu_i(\mathbf{b})$ is replaced by m_i. Likewise, $\mu_i(\boldsymbol{\beta})$ is replaced by μ_i. Derivatives (A.1), (A.3), (A.5) and (A.6) are replaced by (A.1*) to (A.6*) (cf. Appendix 1). Analogously to the model-based case, it can be shown that, as the stratum-specific μ_i correctly model the dissatisfaction probabilities, $\mathbf{m}$ and $\wp$ are asymptotically uncorrelated. Furthermore, the PAD can again be treated in complete analogy to the PAR.

Note that the covariance matrix of $N^{1/2}(\mathbf{m}-\boldsymbol{\mu})$ converges against $\mathrm{diag}_i\{\mu_i(1-\mu_i)/\pi_i\}$. By straightforward transfer, we thus get the variance formula

$$\left(\mathrm{PAR}^{I}\right)^{2}\left[\sum_{\substack{i=0\\ \pi_i>0}}^{2^{L}-1}\frac{\mu_i(1-\mu_i)}{\pi_i}\mathbf{A}_{\mathrm{PAR,mf};m_i}^{2}+\sum_{i=0}^{2^{L}-1}\pi_{i}\mathbf{A}_{\mathrm{PAR,mf},\wp;i}^{2}-\left\{\mathbf{A}_{\mathrm{PAR,mf},\wp}{}^{\mathrm{T}}\boldsymbol{\pi}\right\}^{2}\right] \tag{11}$$

where the derivative vector $\mathbf{A}_{\mathrm{PAR,mf}}$ is given in Appendix 2 (as well as its analogue $\mathbf{A}_{\mathrm{PAD,mf}}$). The asymptotic variance (11) can be consistently estimated in the obvious way.

5 A DATA EXAMPLE

The following example is taken from the so-called quality audit survey (QAS), which is a cross-sectional survey on the satisfaction of customers with the quality of their vehicles for a large group of automotive manufacturers. The data analyzed here come from a combined sample of all manufacturers and all models in the German market. Customers indicate their satisfaction with the vehicle on a ten-point-scale. Dissatisfaction is defined as a satisfaction rating below 7 (1 to 6); it is customary to work with those dichotomized values rather than with the ratings themselves. Furthermore, customers indicate whether or not certain things went wrong on their vehicle. Here, these "Things Gone Wrong" (TGW) have been grouped into four coarse groups. The data are listed in Table 1. Note that binarizing the stratum number provides the pattern of TGW, e.g. i=0 corresponds to 0 0 0 0 (no TGW whatsoever) and i=5 corresponds to 0 1 0 1 (exterior and chassis TGW present, but no interior and powertrain TGWs). It is straightforward to calculate the common attributable risk, which is formula (1) applied with X denoting at least one TGW vs. no TGW whatsoever. This quantity can be estimated as est(AR) = (1403/6896 – 81/1437)/(1403/6896) = 72.3% with a corresponding estimated common attributable difference of est(AD) = 1403/6896–81/1437 = 14.7 percentage points. This means that eliminating all TGWs would make 72.3% of the dissatisfied customers satisfied, which corresponds to a reduction of 14.7 in dissatisfaction percentage points.

Table 1: QAS data on German customers at 36 months in service

Stratum number i	TGWs with: interior	exterior	power-train	chassis	Dis-satisfied customers in stratum i	n_i: Sample size in stratum i	$\bar{y}_i$: Proportion dissatisfied in stratum i
0	no	no	no	no	81	1437	0.0564
1				yes	27	181	0.1492
2			yes	no	73	532	0.1372
3				yes	56	206	0.2718
4		yes	no	no	43	369	0.1165
5				yes	16	135	0.1185
6			yes	no	69	334	0.2066
7				yes	57	203	0.2808
8	yes	no	no	no	39	416	0.0938
9				yes	26	139	0.1871
10			yes	no	86	406	0.2118
11				yes	89	259	0.3436
12		yes	no	no	55	355	0.1549
13				yes	67	283	0.2367
14			yes	no	207	663	0.3122
15				yes	412	978	0.4213
				Total	1403	6896	0.2035

Table 2 shows results from applying the previously described methods to the data from Table 1. The left-most column shows that this example suffers from the problem that individual attributable risks according to (1) sum to almost 140%, which is obviously implausible. The model-free PAR and PAD estimates behave in a much more desirable way: They sum to the common attributable risk of 72.3% and the common attributable difference of 14.7 percentage points, respectively.

Table 2: Attributable risks and attributable differences for the example

	(a)	Model-free estimates (b)		Model-based estimates[3)] (c)		Hybrid estimates: (c) rescaled to (b) total	
	AR[1)] AD[2)]	PAR[1)] PAD[2)]	*Std(PAR)* *Std(PAD)*	PAR[1)] PAD[2)]	*Std(PAR)* *Std(PAD)*	PAR[1)] PAD[2)]	*Std(PAR)*[4)] *Std(PAD)*[4)]
Interior	30.8	15.2	*2.39*	15.6	*2.16*	16.5	*2.28*
	6.1	3.1	*0.49*	3.2	*0.44*	3.4	*0.47*
Exterior	34.0	13.5	*2.4*	11.9	*2.01*	12.6	*2.12*
	6.9	2.8	*0.49*	2.4	*0.41*	2.6	*0.43*
Power-	29.2	28.1	*2.46*	28.3	*2.21*	29.8	*2.33*
train	5.9	5.7	*0.52*	5.8	*0.47*	6.1	*0.49*
Chassis	46.5	15.5	*2.2*	12.6	*1.54*	13.3	*1.63*
	9.5	3.2	*0.45*	2.6	*0.32*	2.7	*0.34*
Total	139.8	72.3		68.5		72.3	
	28.5	14.7		13.9		14.7	

Notes

1) AR and PAR are given in % of the dissatisfied customers.
2) AD and PAD are given in percentage points of the total population.
3) The model-based estimates are based on a logistic main-effects model, i.e. a model according to equation (4) with just an intercept and the explanatory variables Interior (yes/no), Exterior (yes/no), Powertrain (yes/no) and Chassis (yes/no).
4) The standard deviations from the hybrid approach have been determined by the ad-hoc method of rescaling the standard deviations in the same way as the PARs/PADs.

When comparing the model-free PARs and PADs to the ARs and ADs from the first column of Table 2, it is interesting that the ARs and ADs for interior, exterior and chassis grossly overstate the impact of those TGW groups, while the powertrain ARs and ADs are only slightly larger than the corresponding PARs and PADs. This comparison illustrates that a simple rescaling of individually calculated attributable risks is inappropriate: From simple ARs we would conclude that powertrain TGWs are least important, while in fact they are most important.

Next, let us consider the estimated PARs and PADs from a logistic main effects model. These quantities sum up to the model-based common attributable risk of 68.5% and to the model-based common attributable difference of 13.9 percentage points. We see that the model-based estimates are more pessimistic about what can be recovered by eliminating TGWs. This difference is due to the lack of fit from the logistic main effects model. When looking at individual TGW groups, the lack of fit is most apparent for exterior and chassis TGWs.

The standard deviations for the PARs and PADs demonstrate that there is quite some uncertainty in the results, even though only four TGW groups are assessed based on almost 7000 customers (cf. Table 1). The estimated standard deviations from the model-based approach are somewhat smaller than those from the model-free approach, presumably because of fewer parameters in the model.

In this example it has been possible to calculate the model-free estimates. In typical real-world applications, there can be as many as e.g. 19 TGW groups to be simultaneously considered. In such cases, the model-free approach is prohibitive for determining PARs and PADs. Thus, a model-based approach has to be used. Certainly, one has to take precautions to obtain a well-fitting model, in order to avoid bias from model misspecification. The model-free common attributable risk, i.e. the model-free attributable risk for any TGW regardless of category according to (1), can still be calculated even for very large numbers of TGW groups. Hence, the difference between the model-based and the model-free common attributable risk can be used as one means of assessing the fitness of the logistic model for PAR and PAD estimation. In case of a moderate mismatch, one may choose the route of rescaling the model-based PARs and PADs to match the model-free overall sum. For the example, such an approach is demonstrated in the last two columns of Table 2. It has the advantage of removing an implausible overall result. However, distortions in relations between TGW groups cannot be removed by such an approach. Hence, it remains important to find a well-fitting logistic model.

6 DISCUSSION

This paper has provided the asymptotic distribution for the partial attributable risk estimates in cross-sectional studies. Along with the partial attributable risk, the partial attributable difference has also been treated. Although based on a quite elegant piece of logic (cf. e.g. Land and Gefeller 1997), PAR and PAD have so far not been widely used in practice, presumably because they are not implemented in standard software packages and because a variance estimate was not available. Now, with a variance estimate in place, one of the hurdles in introducing the partial attributable risk into research practice has been removed. Computation remains a hurdle for the practitioner, as so far no software package offers PARs or PADs. Programming the method is a very tedious task; we implemented the formulae in the matrix language SAS/IML and checked correctness of our programming by simulations. Computing time strongly depends on the number of possible causes and the number of different occurrence patterns of possible causes.

Now, some areas for further research are outlined: The example has shown that the model-based PAR needs a well-specified logistic model. Thus it is important to be able to enhance model (4) by including regressor variables that are not to be treated as a possible cause but as confounders only. Such an adjusted version of the PAR has not been treated here. It should be straightforward (though tedious) to transfer the variance calculation to the adjusted PAR. A further interesting topic is the performance of the variance estimates for relatively small samples. For small numbers of possible causes, simulation studies are feasible, so that this behavior can be investigated. The very small simulation studies for checking the correctness of our programming – based on 1000 data sets with 1000 observations each, two Y variables and four possible causes – always yielded an average estimated standard deviation that was within 5% of the empirical standard deviation of the estimates. More extensive simulation studies could certainly provide more insight here. In addition, research on the behavior of the model-based estimates for moderately inappropriate logistic models would be of interest. Here, it might also be interesting to investigate whether the finite sample performance of the variance estimate improves by including a consistent estimate for formula (9) rather than working with the limit that assumes correct modeling of the adverse event probabilities. Furthermore, extension of the result to other study types – e.g. case control studies – should be of interest. Also, the method used here is promising for developing the variance formulae for newly proposed risk estimators that exploit hierarchies and grouping among the possible causes (cf. Land et al. 2001a and 2001b).

7 REFERENCES

Basu, S. and Landis, J.R. (1995) Model-based estimation of population-attributable risk under cross-sectional sampling. *American Journal of Epidemiology* **141**, 1338-1343.

Baur, U. (2000) A new effective method for estimating partial attributable risks. Diploma Thesis, Department of Statistics, University of Dortmund, Germany.

Benichou, J. and Gail, M.H. (1989) A delta method for implicitly defined random variables. *The American Statistician* **43**, 41-44.

Cox, L.A. (1985) A new measure of attributable risk for public health applications. *Management Science* **7**, 800-813.

Eide, G.E. and Gefeller, O. (1995) Sequential and average attributable fractions as aids in the selection of preventive strategies. *Journal of Clinical Epidemiology* **48**, 645-655.

Gefeller, O., Land, M. and Eide, G.E. (1998) Averaging attributable fractions in the multifactorial situation: Assumptions and interpretation. *Journal of Clinical Epidemiology* **51**, 437-441.

Grömping, U. and Weimann, U. (2003) A statistical method for customer satisfaction inhibitor estimation that permits calculation of confidence intervals. *Ford Technical Journal* **6**(1) (online).

Kruskal, W. (1987) Relative importance by averaging over orderings. *The American Statistician* **41**, 6-10.

Land, M., Vogel, C. and Gefeller, O. (2001a) A multifactorial variant of the attributable risk for groups of exposure variables. *Biometrical Journal* **43**, 461-481.

Land, M., Vogel, C. and Gefeller, O. (2001b) Partitioning methods for multifactorial risk attribution. *Statistical Methods in Medical Research* **10**, 217-230.

Land, M. and Gefeller, O. (1997) A game-theoretic approach to partitioning attributable risks in epidemiology. *Biometrical Journal* **39**, 777-792.

Walter, S.D. (1980) Prevention for multifactorial diseases. *American Journal of Epidemiology* **112**, 409-416.

Walter, S.D. (1983) Effects of interaction, confounding and observational error on attributable risk estimation. *American Journal of Epidemiology* **117**, 598-604.

Wille, L. and Gefeller, O. (1996) Partitioning the disease risk among several exposure factors: a computational solution to an epidemiological problem. In: *Softstat '95, Advances in Statistical Software 5* (eds. Faulbaum, F. and Badilla, W.), pp. 249-256. Stuttgart: Lucius and Lucius.

APPENDIX 1: CALCULATION OF DERIVATIVES OF g_1 (8a) AND g_2 (8b)

Derivatives w.r.t. the $\bar{y}_i$

$$\left(\frac{\partial g_1}{\partial \bar{y}_i}\right)_{\mathbf{b}=\boldsymbol{\beta},\wp=\boldsymbol{\pi},\mathbf{Y}_N=\mathbf{v}_y} = \pi_i \mathbf{x}_i^{\mathrm{T}} . \tag{A.1}$$

$$\left(\frac{\partial g_2}{\partial \bar{y}_i}\right)_{\mathbf{b}=\boldsymbol{\beta},\wp=\boldsymbol{\pi},\mathbf{Y}_N=\mathbf{v}_y} = \mathbf{0}_{2^L} . \tag{A.2}$$

Derivatives w.r.t. the p_i

For strata with $\pi_i>0$ (we assume that N is large enough that strata with positive π_i are non-empty):

$$\left(\frac{\partial g_1}{\partial p_i}\right)_{\mathbf{b}=\boldsymbol{\beta},\wp=\boldsymbol{\pi},\mathbf{Y}_N=\mathbf{v}_y} = \mathbf{x}_i^{\mathrm{T}} \{\mu_i - \mu_i(\boldsymbol{\beta})\} . \tag{A.3}$$

Note that (A.3) is 0, if the logistic model is correct.

$$\left(\frac{\partial g_{2;i}}{\partial p_i}\right)_{\mathbf{b}=\boldsymbol{\beta},\wp=\boldsymbol{\pi},\mathbf{Y}_N=\mathbf{v}_y} = \frac{1}{\pi_i} . \tag{A.4}$$

Any derivative w.r.t. p_i of the j-th component of the estimating function ($g_{2;j}$) with $j\neq i$ is 0.
For strata with $\pi_i=0$, (A.3) and (A.4) are 0, because p_i is 0 with probability one.

Derivatives w.r.t. the parameter estimates b and $\wp_i$

$$\left(\frac{\mathrm{d}\, g_1}{\mathrm{d}\, \mathbf{b}}\right)_{\mathbf{b}=\boldsymbol{\beta},\wp=\boldsymbol{\pi},\mathbf{Y}_N=\mathbf{v}_y} = -\sum_{i=0}^{2^L-1} \pi_i \mu_i(\boldsymbol{\beta})\{1-\mu_i(\boldsymbol{\beta})\}\mathbf{x}_i^{\mathrm{T}}\mathbf{x}_i . \tag{A.5}$$

For strata with $\pi_i>0$ (we assume that N is large enough that strata with positive π_i are non-empty):

$$\left(\frac{\partial g_{2;i}}{\partial \wp_i}\right)_{\mathbf{b}=\boldsymbol{\beta},\wp=\boldsymbol{\pi},\mathbf{Y}_N=\mathbf{v}_y} = \frac{-1}{\pi_i} . \tag{A.6}$$

Any derivative w.r.t. $\wp_i$ of the j-th component of the estimating function ($g_{2;j}$) with $j\neq i$ is 0.
For strata with $\pi_i=0$, (A.6) is 0, because p_i is 0 with probability one.

Derivatives w.r.t. m for the model-free case (Here, g_1 is defined in (8a*).):

$$\left(\frac{\partial g_1}{\partial \bar{y}_i}\right)_{\mathbf{m}=\boldsymbol{\mu},\wp=\boldsymbol{\pi},\mathbf{Y}_N=\mathbf{v}_y} = \pi_i \tag{A.1*}$$

$$\left(\frac{\partial g_1}{\partial p_i}\right)_{\mathbf{m}=\boldsymbol{\mu},\wp=\boldsymbol{\pi},\mathbf{Y}_N=\mathbf{v}_y} = \mu_i - \mu_i = 0 . \tag{A.3*}$$

$$\left(\frac{\mathrm{d}\, g_1}{\mathrm{d}\, \mathbf{m}}\right)_{\mathbf{m}=\boldsymbol{\mu},\wp=\boldsymbol{\pi},\mathbf{Y}_N=\mathbf{v}_y} = \underset{i=0,\dots,2^L-1}{diag}\left(-\pi_i\right) \tag{A.5*}$$

$$\left(\frac{\mathrm{d}\, g_2}{\mathrm{d}\, \mathbf{m}}\right)_{\mathbf{m}=\boldsymbol{\mu},\wp=\boldsymbol{\pi},\mathbf{Y}_N=\mathbf{v}_y} = \mathbf{0}_{2^L\times 2^L} \tag{A.6*}$$

APPENDIX 2: CALCULATION OF THE VECTORS A

The vectors A for the model-based approach

Let Var($Y|C_i$)=P($Y|C_i$){1−P($Y|C_i$)}=$\mu_i(\boldsymbol{\beta})\{1-\mu_i(\boldsymbol{\beta})\}$ with $\mu_i(\boldsymbol{\beta})$ as defined in (4). The derivative of est{P($Y|C_i$)} w.r.t. **b** (evaluated at $\boldsymbol{\beta}$) is given by $\mathbf{x}_i^{\mathrm{T}}$Var($Y|C_i$). This is the key simplification needed for writing down **A**. Now, let $\mathbf{x}_{0,i^\circ j}$ and $\mathbf{x}_{1,i^\circ j}$ denote the row vectors of regressor values for the logistic regression in stratum $C_i^{L-1} \circ C_j^{L-1}$ in combination with X_1=0 and X_1=1 respectively. In the following, the elements for the derivative are spelled out.

Derivatives w.r.t. the logistic regression parameter vector b

$\mathbf{A}_{\mathrm{PAR,mod;}\mathbf{b}}$=d ln{est(PAR^l)}/d**b**

$$= -\,\mathrm{d}\ln[\mathrm{est}\{\mathrm{P}(Y{=}1)\}]/\mathrm{d}\mathbf{b}$$

$$+\,\mathrm{d}/\mathrm{d}\mathbf{b}\left[\ln\left[\mathrm{est}\left[\sum_{i=0}^{2^{L-1}-1} P\left(C_i^{L-1}, X_1=1\right)\sum_{j=0}^{2^{L-1}-1}\left(L-j^{+}-1\right)!\left(j^{+}\right)!\left\{P(Y=1\mid C_i^{L-1}\circ C_j^{L-1}, X_1=1)-P(Y=1\mid C_i^{L-1}\circ C_j^{L-1}, X_1=0)\right\}\right]\right]\right]$$

$$= \quad -\frac{\sum_{i=0}^{2^{L}-1} P(C_i)\dfrac{\mathrm{d}}{\mathrm{d}\mathbf{b}}\mathrm{est}\{P(Y=1\mid C_i)\}}{P(Y=1)}$$

$$+\;\frac{\sum_{i=0}^{2^{L-1}-1} P\left(C_i^{L-1}, X_1=1\right)\sum_{j=0}^{2^{L-1}-1}\left[\left(L-j^{+}-1\right)!\left(j^{+}\right)!\left[\dfrac{\mathrm{d}}{\mathrm{d}\mathbf{b}}\mathrm{est}\{P(Y=1\mid C_i^{L-1}\circ C_j^{L-1}, X_1=1)\}-\dfrac{\mathrm{d}}{\mathrm{d}\mathbf{b}}\mathrm{est}\{P(Y=1\mid C_i^{L-1}\circ C_j^{L-1}, X_1=0)\}\right]\right]}{\sum_{i=0}^{2^{L-1}-1} P\left(C_i^{L-1}, X_1=1\right)\sum_{j=0}^{2^{L-1}-1}\left(L-j^{+}-1\right)!\left(j^{+}\right)!\left\{P(Y=1\mid C_i^{L-1}\circ C_j^{L-1}, X_1=1)-P(Y=1\mid C_i^{L-1}\circ C_j^{L-1}, X_1=0)\right\}}$$

$$= -\frac{\sum_{i=0}^{2^{L}-1} P(C_i)\mathbf{x}_i^{\mathrm{T}} Var(Y=1\mid C_i)}{P(Y=1)} \tag{A.7}$$

$$+\;\frac{\sum_{i=0}^{2^{L-1}-1} P\left(C_i^{L-1}, X_1=1\right)\sum_{j=0}^{2^{L-1}-1}\left(L-j^{+}-1\right)!\left(j^{+}\right)!\left\{\mathbf{x}_{1,i\circ j}^{\mathrm{T}}\, Var(Y=1\mid C_i^{L-1}\circ C_j^{L-1}, X_1=1)-\mathbf{x}_{0,i\circ j}^{\mathrm{T}}\, Var(Y=1\mid C_i^{L-1}\circ C_j^{L-1}, X_1=0)\right\}}{\sum_{i=0}^{2^{L-1}-1} P\left(C_i^{L-1}, X_1=1\right)\sum_{j=0}^{2^{L-1}-1}\left(L-j^{+}-1\right)!\left(j^{+}\right)!\left\{P(Y=1\mid C_i^{L-1}\circ C_j^{L-1}, X_1=1)-P(Y=1\mid C_i^{L-1}\circ C_j^{L-1}, X_1=0)\right\}}$$

The derivative of ln{est(PAD^l)} is simpler, as the first summand can be dropped.

$\mathbf{A}_{\text{PAD,mod;}\mathbf{b}}$ is thus given as (A.7) without the first summand. (A.8)

Derivatives w.r.t. $\wp_k$, for $k < 2^{L-1}$, i.e. for strata with X_1=0

These stratum probabilities occur in the first summand of PAR only, and they don't occur at all in PAD. Thus,

$\mathbf{A}_{\text{PAR,mod;}\wp_k} = \partial \ln\{\text{est}(\text{PAR}^l)\}/\partial \wp_k = -\partial \ln[\text{est}\{P(Y=1)\}]/\partial \wp_k = -P(Y=1|C_k)/P(Y=1)$, (A.9)

$\mathbf{A}_{\text{PAD,mod;}\wp_k} = 0$. (A.10)

Derivatives w.r.t. $\wp_k$, $k=2^{L-1},\ldots,2^L-1$, i.e. with X_1=1

These stratum probabilities occur in both summands of PAR and in the only summand of PAD. Thus,

$\mathbf{A}_{\text{PAR,mod;}\wp_k} = \partial \ln\{\text{est}(\text{PAR}^l)\}/\partial \wp_k$ (A.11)

$$= -\partial \ln[\text{est}\{P(Y=1)\}]/\partial \wp_k$$

$$+\,\partial/\partial \wp_k \left[\ln\left[\text{est}\left[\sum_{i=0}^{2^{L-1}-1} P\left(C_i^{L-1}, X_1=1\right) \sum_{j=0}^{2^{L-1}-1} \left(L-j^+-1\right)!\left(j^+\right)!\left\{P(Y=1|C_i^{L-1}\circ C_j^{L-1}, X_1=1) - P(Y=1|C_i^{L-1}\circ C_j^{L-1}, X_1=0)\right\}\right]\right]\right].$$

$$= -P(Y=1|C_k)/P(Y=1) + \frac{\sum_{j=0}^{2^{L-1}-1} \left(L-j^+-1\right)!\left(j^+\right)!\left\{P(Y=1|C_{k-2^{L-1}}^{L-1}\circ C_j^{L-1}, X_1=1) - P(Y=1|C_{k-2^{L-1}}^{L-1}\circ C_j^{L-1}, X_1=0)\right\}}{\sum_{i=0}^{2^{L-1}-1} P\left(C_i^{L-1}, X_1=1\right) \sum_{j=0}^{2^{L-1}-1} \left(L-j^+-1\right)!\left(j^+\right)!\left\{P(Y=1|C_i^{L-1}\circ C_j^{L-1}, X_1=1) - P(Y=1|C_i^{L-1}\circ C_j^{L-1}, X_1=0)\right\}}.$$

$\mathbf{A}_{\text{PAD,mod;}\wp_k}$ is obtained by omitting the first summand from (A.11). (A.12)

Modifications for the model-free approach

(A.7) and (A.8) need modification for the model-free approach. In fact,

$\mathbf{A}_{\mathrm{PAR,mf;}\mathbf{m}}$=d ln{est($\mathrm{PAR}^{l}$)}/d**m**

$$= -\,\mathrm{d}\ \ln[\mathrm{est}\{\mathrm{P}(Y{=}1)\}]/\mathrm{d}\mathbf{m}$$

$$+\ \mathrm{d}/\mathrm{d}\mathbf{m}\left[\ln\left[\mathrm{est}\left[\sum_{i=0}^{2^{L-1}-1} P\left(C_i^{L-1}, X_1=1\right)\sum_{j=0}^{2^{L-1}-1}\left(L-j^+-1\right)!\left(j^+\right)!\left\{P(Y=1\mid C_i^{L-1}\circ C_j^{L-1}, X_1=1)-P(Y=1\mid C_i^{L-1}\circ C_j^{L-1}, X_1=0)\right\}\right]\right]\right]$$

$$=\left(-\frac{P(C_s)}{P(Y=1)}\right)_{s=0,\dots,2^L-1}$$

$$+\left[\frac{\sum_{i=0}^{2^{L-1}-1} P\left(C_i^{L-1}, X_1=1\right)\sum_{\substack{j=0\\ j:C_s=(C_i^{L-1}\circ C_j^{L-1},X_1=1)}}^{2^{L-1}-1}\left(L-j^+-1\right)!\left(j^+\right)!}{\sum_{i=0}^{2^{L-1}-1} P\left(C_i^{L-1}, X_1=1\right)\sum_{j=0}^{2^{L-1}-1}\left(L-j^+-1\right)!\left(j^+\right)!\left\{P(Y=1\mid C_i^{L-1}\circ C_j^{L-1}, X_1=1)-P(Y=1\mid C_i^{L-1}\circ C_j^{L-1}, X_1=0)\right\}}\right]_{s=0,\dots,2^L-1}$$

$$-\left[\frac{\sum_{i=0}^{2^{L-1}-1} P\left(C_i^{L-1}, X_1=1\right)\sum_{\substack{j=0\\ j:C_s=(C_i^{L-1}\circ C_j^{L-1},X_1=0)}}^{2^{L-1}-1}\left(L-j^+-1\right)!\left(j^+\right)!}{\sum_{i=0}^{2^{L-1}-1} P\left(C_i^{L-1}, X_1=1\right)\sum_{j=0}^{2^{L-1}-1}\left(L-j^+-1\right)!\left(j^+\right)!\left\{P(Y=1\mid C_i^{L-1}\circ C_j^{L-1}, X_1=1)-P(Y=1\mid C_i^{L-1}\circ C_j^{L-1}, X_1=0)\right\}}\right]_{s=0,\dots,2^L-1} \quad \text{(A.13)}$$

Again, the corresponding matrix for PAD is obtained by omitting the first summand in (A.13). (A.14)